Philosophy 134
Module 1
Informal Introduction to Modal Logic

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Contents

1 Motivation for Modal Logic

One of the first things learned by beginning logic students is the definition of a *valid argument*. The standard definition of a valid argument runs along these lines: An argument from a set of premises to a conclusion is valid if and only if it is not *possible* for all the premises of the argument to be true and for the conclusion to be false. Alternatively, one might say that an argument is valid just in case it is that *necessarily*, if the premises are true, then the conclusion is true.

The concept of a valid argument lies at the heart of logic. The definition of the concept of a valid argument, in turn, depends essentially on the *modal* concepts of possibility and necessity. These concepts are called "modal" because they indicate a "way" or "mode" in which the truth-values of the premises are connected with the truth-values of the conclusion.

One of the main tasks of symbolic logic is to represent the form of arguments, in such a way that their validity or invalidity can be determined using standardized techniques. One can use truth-tables, for example, to represent the validity or invalidity of arguments whose basic units are individual sentences.

In the logics that are generally taught in introductory logic courses, the properties of validity or invalidity are not represented in the symbolic language itself. There are symbols representing the "truth-functional" operators 'and,' 'or,' 'not,' etc., but there is no symbol in the language for 'therefore.'[1] One might see the symbol '∴,' but this is an expression of the meta-language. In fact, the operators of sentential and predicate logic do not represent possibility or necessity at all. Standard logic is non-modal in this respect, even though it might be used to establish modal properties of arguments.

Modal logics are precisely those logical systems which contain modal operators. In the case of validity one might seek to build a logical language which contains an operator which is understood to express the modal property of validity. That is, it would contain a *modal operator*.

2 The Lewis Systems

Modern modal logic appeared in the early twentieth century, not long after modern non-modal logics had been popularized by Russell and Whitehead's *Principia Mathematica*.[2] A young philosophy instructor at UC Berkeley, C.I. Lewis, used *Principia* as a text. Lewis thought that Russell's description of the truth-functional conditional operator as "material implication" was misleading. In response, he built several axiomatic systems with various modal operators, including one for impossibility and another for consistency. The most interesting of the operators was one he called "strict implication," which Lewis thought better represents the relation between premise and conclusion in an argument than does "material implication." After several false starts, Lewis produced a system that he later called "S3."[3]

One-place modal operators for possibility and necessity eventually became components of the Lewis systems. The necessity operator can be understood as allowing the represention of the concept of logically necessary truth. "Strict implication" in S3 is equivalent to a necessarily true "material implication." The "strict implication" operator and its relation to "material implication" will be described in more detail in Module 6.

Some time after the introduction of the original systems of modal logic, one of which he came to call "S3," Lewis formulated several related systems of modal logic. Systems S4 and S5 were stronger than S3, in that more modal sentences could be proved in them. Systems S1 and S2 are weaker, in that some sentences that could be proved in S3 could not be proved in S1 or S2. In this text, we will be examining in detail only the strongest Lewis systems, S4 and S5. These systems belong to a family of "normal" systems which have the same basic semantics.

3 Matrix Semantics

Lewis's systems were laid down in axiomatic form. The axioms were taken to express basic facts about modality given an intuitive way of understanding what they meant. The earliest work on the systems was to prove theorems of the given systems which follow from their specific axioms. Very soon thereafter, the axiomatic systems were given interpretations. The most prominent kind of interpretation was with "matrices" that resemble truth-tables. A useful matrix for modal logic typically contained more than two values. The following is an example of a matrix for the strict implication symbol "⥽." The numbers 1 and

[1]Many logic students render the "material conditional" into English as 'therefore,' but it will be seen in the next module that this is a mistake.

[2]The basic system in that book had been laid out by Frege in 1879, but had gone largely unnoticed.

[3]The system of *A Survey of Symbolic Logic*, 1918, Chapter V, contained a devastating error in the formulation of one of the axioms. As a result, this chapter appears only in the first edition. The stable version of S3 is given in "Strict Implication—an Emendation," *Journal of Philosophy* Vol. 17 (1920), pp. 300-302.

2 are “designated values,” which play the same role as “truth” in a truth-table, while the numbers 3 and 4 correspond to truth-table “falsehood.”[4] Three rows of interest are highlighted.

A	B	$A \prec B$
1	1	2
1	**2**	**4**
1	3	3
1	4	4
2	1	2
2	2	2
2	3	3
2	4	3
3	1	2
3	**2**	**4**
3	3	2
3	**4**	**4**
4	1	2
4	2	2
4	3	2
4	4	2

A brief examination of the matrix reveals differences between “strict” and “material” implication. In the second and tenth rows, B has the designated value, but $A \prec B$ does not. In the tenth and twelfth rows, A does not have the designated value, and $A \prec B$ does not. With “material implication,” if B has the designated value (truth) or A does not have it (falsehood), then $A \supset B$ has the designated value (truth). Note, however, that there is no row of the table where A has a designated value, B has a non-designated value, and $A \prec B$ has a designated value. “Strict implication” respects a necessary condition of any implication, that it may not take us from “truth” to “falsehood.”

4 Possible-Worlds Semantics

Using matrices, logicians were able to get other important results about the systems. They could determine which systems contain which other systems and whether a given axiom is independent of the other axioms.[5] While the matrix technique was useful, it provided no real insight into the meanings of modal sentences. It was Rudolf Carnap, writing in the mid 1940s, who first provided an intuitively plausible semantics for one of the Lewis systems, S5.[6] In Carnap’s semantical system, the truth-values of non-modal sentences are determined just as they are in truth-functional logic. A sentence whose main operator is a necessity operator is true if and only if the sentence it governs is a logical truth. Thus, if a sentence is true on all rows of its truth-table, then the sentence formed by prefixing a necessity operator to it is also true, and in fact is true on all rows of all truth-tables.

[4] C. I. Lewis and C. H. Langford, *Symbolic Logic* 1932, Appendix II. This matrix is entitled “Group V.”

[5] System S contains system S′ just in case all the theorems of S′ are theorems of S. S is then said to be stronger than S′ and S′ weaker than S. An axiom is independent of the other axioms in an axiom set just in case it cannot be proved as a theorem from the other axioms.

[6] “Modalities and Quantification,” *Journal of Symbolic Logic*, Vol. 11 (1946), pp. 33-64. See also *Meaning and Necessity*, 1947.

Carnap also provided a system for a predicate-logic version of S5. His semantics is of interest because of the way it interprets the syntax of predicate logic. Non-modal semantics interprets terms as standing for objects in a universe or domain of discourse. Predicates are interpreted as standing for sets of objects from the domain. Beside this "extensional" type of interpretation, Carnap developed an "intensional" interpretation suitable for the use of terms and predicates in modal contexts.

The intensional interpretation depends on the notion of a "state description". Carnap wrote that "the state description represents Leibniz's 'possible worlds' or Wittgenstein's possible states of affairs."[7] A term might stand for different objects in different state descriptions, so that its "intension" is a function from state descriptions to objects. A predicate might have different extensions in different state descriptions, so that its "intension" is a function from state descriptions to sets of objects.

In the late 1950s, Carnap's semantics was generalized to the form in which it exists today.[8] The key notion in the semantics is that of a "possible world." In sentential logic, a possible world corresponds to a row of a truth-table. In predicate logic, a possible world corresponds (roughly) to an interpretation, which spells out what objects there are and what properties they have.

Carnap's system, in effect, took necessity to be truth at all possible worlds. This worked as a semantics for S5, but not for any of the weaker systems of Lewis and others. The innovation was to add to the semantics a two-place relation of "accessiblity" or "alternativeness" holding among the worlds themselves. Then a necessity sentence could be taken to be true just in case the sentence governed by the necessity operator is true at all accessible possible worlds. This generalization of the Carnapian semantics allowed Kripke and others to provide semantics for most of the known axiomatic systems of modal logic. It also made it easy to generate new systems. Most importantly, perhaps, it provided an intuitive way of understanding what the sentences of modal logic mean.

It should be noted that for a long time, modal logic was held in some disrepute, due to the criticisms of W.V.O. Quine.[9] One of his objections was that valdity, implication, logical necessity, etc., are meta-logical notions which have no place in logic itself. Another was that the semantics for modal predicate logic requires the postulation of possible but non-existent objects. Quine believed that we should not commit ourselves to "possibilia," on the grounds that they do not have well-defined "identity conditions."

When generalized possible worlds semantics came on the scene, philosophers welcomed it as a powerful analytical tool and brushed Quine's objections aside. It is probably not coincidental that about this time there was a powerful shift away from the austere ontologies of Quine and his Harvard colleague Nelson Goodman (not to mention the later Wittgenstein) to ontologies which allow the existence of possible, but not actual, objects. There remains vigorous debate about the metaphysical status of possible worlds and objects in them. At one extreme, David Lewis advocated a modal realism, according to which each possible world is just as real as the one we call "actual."[10] At another, Michael Jubien has tried to treat modalities without appeal to possible worlds at all.[11]

[7] *Meaning and Necessity*, 9-10.

[8] The sematics is commonly attributed to Saul A. Kripke, but it was developed during the same period by Jaakko Hintikka and Stig Kanger.

[9] See, for example, "Three Grades of Modal Involvement," *Proceedings of the XIth International Congress of Philosophy*, Brussels, 1953, Vol. 14, reprinted in Quine's *The Ways of Paradox* (1966).

[10] *On the Plurality of Worlds*, 1986.

[11] *Contemporary Metaphysics*, 1997, Chapter 8.

5 Applications

From the time generalized possible worlds semantics was invented, and even before, philosophical logicians began to recognize that it has applications beyond the logical notions of implication and logical truth. Jaakko Hintikka recognized that Lewis's "necessity" operator could be interpreted either as a belief operator or a knowledge operator.[12] Arthur Prior had explored the use of the modal operators to represent modalities of time.[13] G. H. von Wright in 1951 had interpreted the modal operators in terms of obligation and permissibility.[14] All of these suggestions have been developed in great detail during the past few decades.

6 Natural Deduction Systems

In 1952, Frederick Fitch published a very influential logic text, *Symbolic Logic: An Introduction*. In Chapter 3 of this book, Fitch adapted the "natural deduction" systems of Gerhard Gentzen and others in a way that made it relatively easy to prove theorems and the validity of arguments syntatically. Fitch also provided rules for the Lewis system S4 and a system "almost the same as" S2.[15] Fitch's approach has been generalized to a number of other modal systems.

7 Plan of the Text

The aim of this text is twofold. The first aim is to acquaint the reader with the basic formal characteristics of a wide range of systems of modal logic. Each system will be introduced semantically. Fitch-style natural deduction rules will then be given and treated as shorthand for obtaining semantical results. Axiomatic formulations of the various systems will also be given.

The text begins with a review of non-modal sentential logic. Then, after the introduction of a basic symbolic language of modal logic, a way of interpreting it, and rules of inference applying to it, a basic sentential modal system, K, will be laid out. Then, a number of other sentential systems will be considered. Finally, some systems of predicate modal logic will be introduced.

The second aim of the text is to explore the ways in which the various systems can be applied. We will consider modality in six different ways: (1) as a logic of necessary truth and falsehood, "alethic" modal logic; (2) as a logic of implication between a sentence and what follows from it, "implicational" modal logic; (3) as a logic of obligation and permission, "deontic" logic; (4) as a logic of knowledge, "epistemic" logic; (5) as a logic of belief, "doxastic" logic; (6) as a logic of time, "temporal logic."

[12] *Knowledge and Belief: An Introduction to the Logic of the Two Notions*, 1962.
[13] *Time and Modality*, 1957.
[14] "Deontic Logic," in *Mind*.
[15] These rules map very nicely onto the generalized possible worlds semantics, as will be seen. Fitch did not recognize the connection, however.